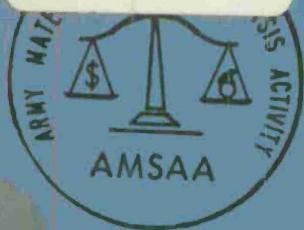


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AN APPROACH TO FIRE CONTROL SYSTEM COMPUTATIONS AND SIMULATIONS

JAMES F. LEATHRUM

APRIL 1975

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U. S. ARMY MATERIEL SYSTEMS ANALYSIS ACTIVITY
Aberdeen Proving Ground, Maryland

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FOREWORD

This report documents work performed by Dr. J. F. Leathrum of the University of Delaware in his capacity as a consultant to the System Simulation & Support Branch, Combat Support Division of AMSAA.

The author would like to acknowledge the assistance of Mr. T. R. Perkins who has programmed many of the algorithms, assisted in the preparation of and proofread the manuscript and Mr. H. H. Burke, who provided the numerous open literature publications which served to motivate the writing of this report.

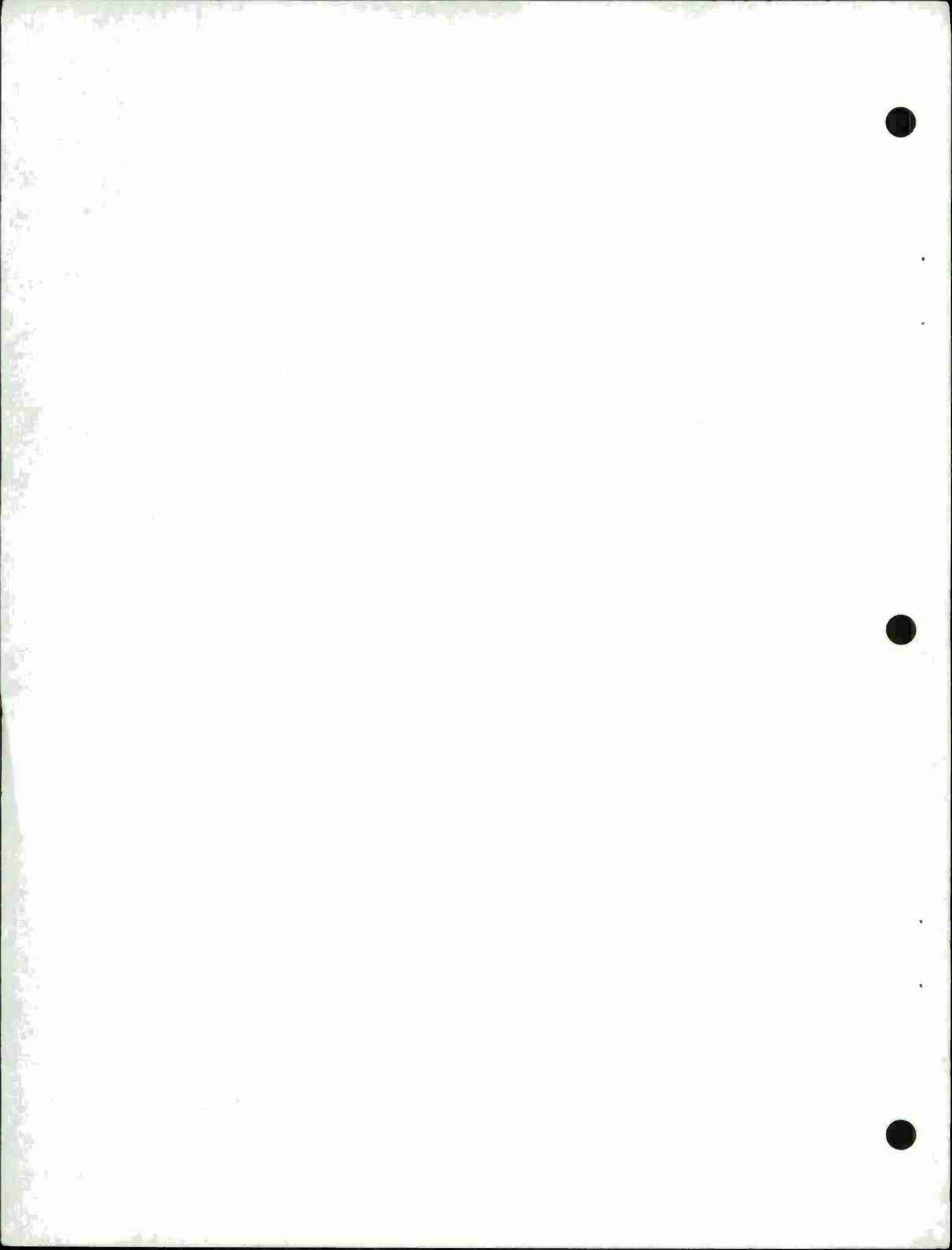
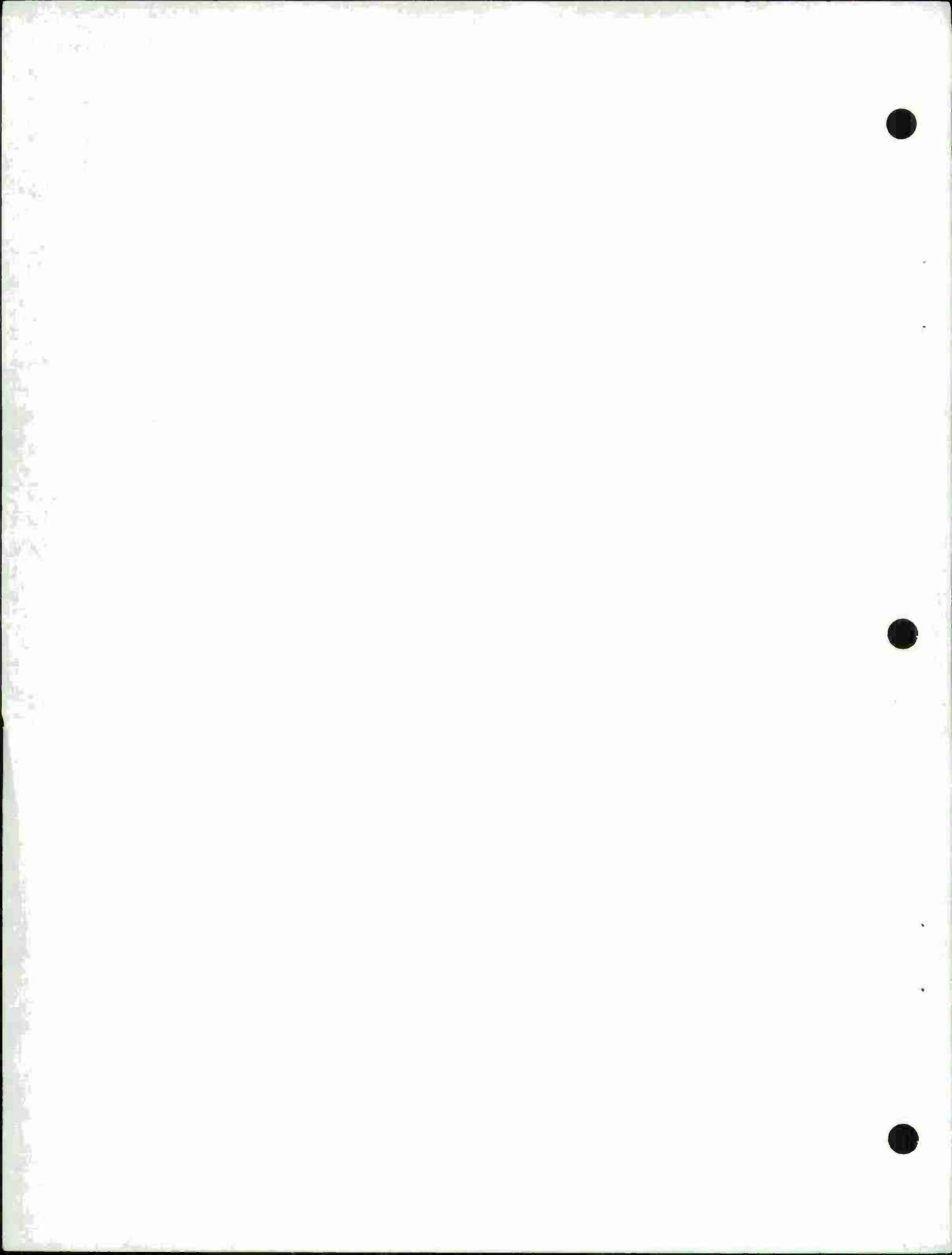


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GLOSSARY OF SYMBOLS

a	Linear acceleration
A	System plant matrix
b	Input matrix
B	Input matrix
G	Control gain matrix
H	Observation matrix
I	Identity matrix
J	Performance criterion
K	Kalman gain or control influence matrix
M	Transformation matrix
O	Null matrix
P	Covariance matrix
p	Matrix dimension
q	Matrix dimension
Q	Input noise variance
Q*	Transformed noise variance
R	Observation noise variance
r	Slant range
T	Transformation matrix
t	Time
t_f	Projectile time of flight
Δt	Incremental time
U	System input vector
V	Observation noise vector

v	Linear velocity
w	Input vector
x	State vector
x,y,z	Position coordinates
Δx	Error vector
Y	Observation vector
Z	Biased estimator matrix
α	Reciprocal of correlation time constant
γ	Matrix subset
θ	Elevation angle
ψ	Azimuth angle
ϕ	Transition matrix
λ	Weighting matrix
Γ	Weighting matrix
ω	Angular velocity

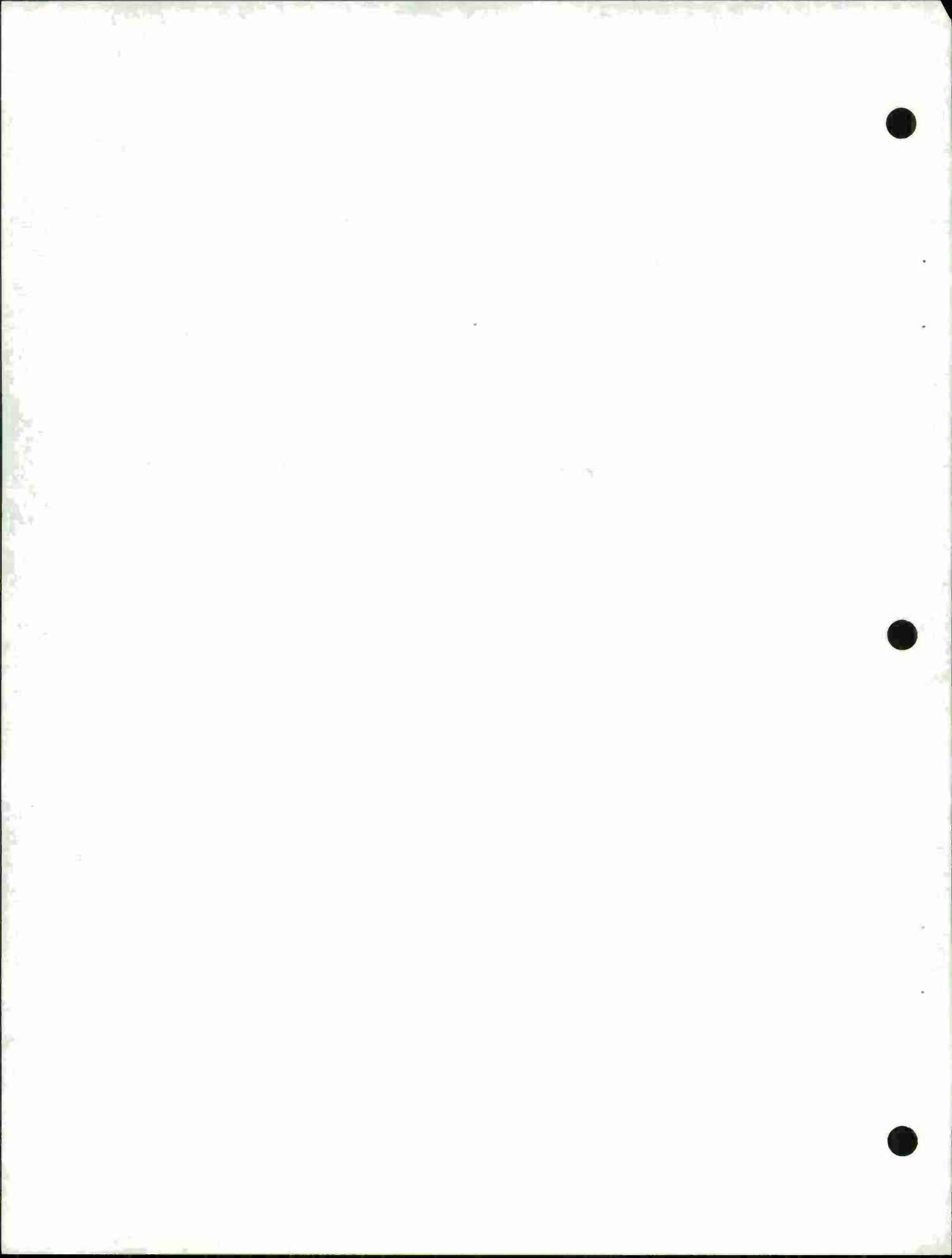
Subscripts

E	Earth reference frame
i,j	Indices
k	Index of time
L	Center of curvature in earth frame
LOS	Line of sight
N	Index
o	Initial condition
p,q	Dimension of identity matrix
r	Spherical coordinate frame

rE Spherical to earth
SE Sensor to earth
TE Target to earth
v Velocity reference frame
vE Velocity to earth
z,y,z Coordinate reference frame

Superscripts

c Center of curvature reference
k Index of time
T Matrix transpose
S Sensor



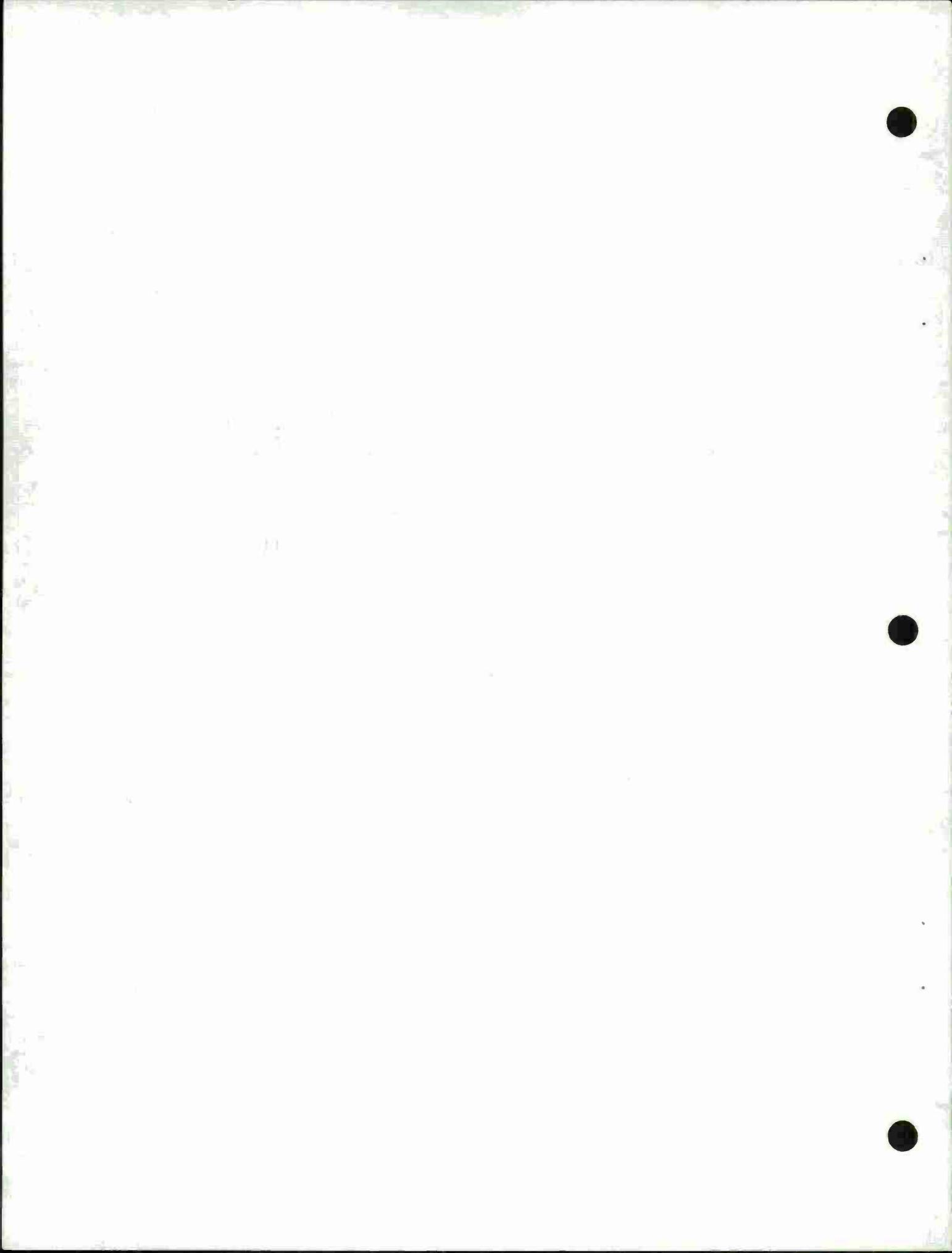
INTRODUCTION

The purpose of this work is to provide a tutorial discussion of the important computational problems associated with modern fire control systems. A variety of such systems has been proposed, all of which employ estimators and digital computers in one form or another. Although there is some consistency in the objectives of the proposed systems, there is considerable variety in the approach used in implementing them. This discussion will consider the various approaches to develop estimators for fire control systems.

In the course of developing this report, a fire control simulation facility is established and tested. This simulation facility is a highly structured set of estimation algorithms and their supporting software together with very high level language and data editing features. It is possible using such a software package to evaluate fire control systems for the many proposed configurations.

In the process of examining the important features of modern fire control systems, a new approach to the target state estimation problem has been developed. In a single state transition model, the target positions are represented in spherical coordinates, and the velocities and accelerations are represented in Cartesian coordinates.

The advantages of such an approach arise in computational considerations. Unnecessary coordinate transformations are eliminated, and the approximations due to linearization of the target and observation models are kept to a minimum.



1. Models

Modern fire control systems which employ "optimal" estimators of the target state require hypothetical mathematical models as an internal representation of the target behavior. The models include an observation model and a target dynamics model. The observation model states the relationship between the actual target state and the observed state. The observation model usually acknowledges the existence of sensor errors. The magnitudes of sensor errors depend directly upon the accuracy with which the sensor is pointed at the target, i.e. The target must be kept within the field of view of the sensor. The problem of pointing the sensor is a classical regulator problem whose solution will require a model of the sensor. These three models of the target, observations, and sensor will be developed in the sections which follow.

1.1 Target Model

In the Kalman style of target state estimators, it is always important to hypothesize a model for the motion of the target. Ideally, this model should be based upon Newtonian mechanics and reflect the actual behavior of the target. In fact, there is not enough information about targets to substantiate a Newtonian model. As a result the accelerations are usually modeled as a random process. The resulting model of target dynamics is:

(a) Position dynamics

$$\dot{\underline{x}} = \underline{v}; \quad \underline{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad 1.1$$

(b) Velocity dynamics

$$\dot{\underline{v}} = \underline{a}; \quad \underline{v} = \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} \quad 1.2$$

(c) Correlated random accelerations

$$\dot{\underline{a}} = -\alpha \underline{a} + \underline{b} \underline{U}; \quad \underline{a} = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} \quad 1.3$$

where \underline{U} is assumed to be a noise process. The combined states of position, velocity, and acceleration may be included in a new state vector, \underline{x} .

$$\underline{x} \equiv \begin{bmatrix} x \\ \dot{x} \\ a_x \\ y \\ \dot{y} \\ a_y \\ z \\ \dot{z} \\ a_z \end{bmatrix} \quad 1.4$$

which leads to a form

$$\dot{\underline{x}} = \underline{A} \underline{x} + \underline{B} \underline{U} \quad 1.5$$

where \underline{A} and \underline{B} are functions of α and \underline{b} .

Assuming that \underline{U} is constant over a small time interval, Δt , Equations 1.5 may be integrated over Δt . The result is a discrete model.

$$\underline{x}_{k+1} = \underline{\Phi}_{k-k} \underline{x}_k + \underline{B}_{k-k} \underline{U}_k \quad 1.6$$

where the parameter, k , is the total number of time steps Δt , to have passed. The matrices, $\underline{\Phi}_k$, and \underline{B}_k , are functions of Δt , and \underline{b} .

Target models of the form of Equation 1.6 predominate in the design of target state estimators. In such applications the noise, \underline{U}_k , is usually assumed to be "white" in the sense that it is uncorrelated with noise which occurred at any other time.

1.2 Observation Model

The observations of target motion are usually incomplete in the same sense that only position and perhaps a few rates are observed. The observations are modeled as a linear combination (usually a linear selection) of the target state components which are corrupted by a measurement error, \underline{v}_k .

$$\underline{Y}_k = \underline{H} \underline{X}_k + \underline{V}_k$$

1.7

The vector, \underline{Y}_k , is the perceived target motion.

Most of the current generation of fire control systems employ a target model of the form of Equations 1.6 and an observation model of the form of Equations 1.7. The observation errors, \underline{V}_k , are also modelled as white noise which are further uncorrelated with the target noise, \underline{U}_k .

1.3 Models of Noise

Given a target model of the form developed in the previous section, the question arises as to how to use the model to design an estimation algorithm which will both estimate and predict the state of the target. The latest generation of target state estimators utilize "optimal" estimates and an algorithm due to Kalman. This algorithm requires adoption of a statistical model of the noise vectors, i.e.

$$E(\underline{U}_k) = E(\underline{V}_k) = 0 \quad 1.8$$

$$E(\underline{U}_k \underline{U}_k^T) = \underline{Q}_k \quad 1.9$$

$$E(\underline{V}_k \underline{V}_k^T) = \underline{R}_k \quad 1.10$$

$$E(\underline{U}_k \underline{V}_k^T) = 0 \quad 1.11$$

as well as the assumptions of whiteness, i.e.

$$E(\underline{U}_j \underline{U}_k^T) = \underline{0} \text{ for } j \neq k \quad 1.12$$

$$E(\underline{U}_j \underline{V}_k^T) = \underline{0} \text{ for } j \neq k \quad 1.13$$

The matrices, \underline{Q}_k and \underline{R}_k , are important parameters in the estimation algorithms to be considered. Usually \underline{R}_k can be fixed by knowledge of sensor accuracy. The choice of \underline{Q}_k is much more subtle since the actual target has no such parameter associated with it. The parameters, α (Equation 1.3) and \underline{Q} , must be chosen in such a way that the performance

of the estimators is optimized. This usually requires consideration of the magnitude of prediction errors which are the ultimate performance measures for a fire control system.

1.4 Sensor Model and Optimal Tracking

In order to compute the estimates suggested in the previous section, a sensor must be available to provide the inputs, \underline{Y} . This sensor must be pointed at the target to whatever accuracy is required to keep the target within the "field of view" of the sensor. A control problem emerges at this point in order to determine what the commands to the sensor should be to keep it pointing at the target. The control action may be determined from the models:

(a) Target Model

$$\underline{x}_{k+1} = \underline{\Phi}_k \underline{x}_k + \underline{B}_k \underline{U}_k \quad 1.14$$

(b) Sensor Model

$$\underline{x}_{k+1}^S = \underline{\Phi}_k \underline{x}_k^S + \underline{K}_k \underline{W}_k \quad 1.15$$

and a performance criterion

$$J = \underline{\Delta x}_N^T \underline{\lambda}_N \underline{\Delta x}_N + \sum_{i=0}^{N-1} [\underline{\Delta x}_i^T \underline{\lambda}_i \underline{\Delta x}_i + \underline{W}_i^T \underline{\Gamma}_i \underline{W}_i] \quad 1.16$$

where

$\underline{\lambda}_i, \underline{\Gamma}_i$ ≡ Weighting matrices

$\underline{\Delta x}_i \equiv \underline{x}_i - \underline{x}_i^S$

\underline{K} , Control Influence Matrix

\underline{W}_k , Command to the sensor

Assuming for convenience that $\underline{\Phi}_k = \underline{\Phi}_k^S$, the models may be combined into

$$\underline{\Delta x}_{k+1} = \underline{\Phi}_k \underline{\Delta x}_k - \underline{K}_k \underline{W}_k + \underline{B}_k \underline{U}_k \quad 1.17$$

The stochastic, optimal control problem of Equations 1.16 and 1.17 is known to have a solution of the form,

$$\underline{w}_k = \underline{G}_k \Delta \hat{\underline{x}}_k$$

1.18

where $\Delta \hat{\underline{x}}_k$ is the best linear unbiased estimate of $\Delta \underline{x}$ obtained by ignoring the control term, $\underline{K}_k \underline{w}_k$.

The control gain, \underline{G}_k , is obtained from

$$\underline{G}_k = (\underline{\Gamma}_k + \underline{K}_k^T \underline{P}_{k+1} \underline{K}_k)^{-1} \underline{K}_k^T \underline{P}_{k+1} \underline{\Phi}_k \quad 1.19$$

where the matrix, \underline{P}_{k+1} , comes from a reverse recursion

$$\underline{P}_k = \underline{\Phi}_k^T (\underline{P}_{k+1}^{-1} + \underline{K}_k \underline{\Gamma}_k \underline{K}_k^T)^{-1} \underline{\Phi}_k + \underline{\lambda}_k \quad 1.20$$

$$\underline{P}_N = \underline{\lambda}_N \quad 1.21$$

With fixed values of $\underline{\Phi}_k$, $\underline{\Gamma}_k$, $\underline{\lambda}_k$, and \underline{K}_k , a fixed value of \underline{G}_k at steady state may be found from setting $\underline{P}_k = \underline{P}_{k+1}$ in equation 1.20 and solving Equation 1.21 for \underline{P}_k .

The optimal control problem of Equations 1.16 and 1.17 arises again in the gun pointing problem. In this case miss distances may be fed back through an optimal gain to produce a command to the gun.

2. LINEARIZATION

In all of the discussions in the previous sections, it was implicitly assumed that every term of any equation was expressed in the same coordinate system. The question immediately arises concerning what that coordinate system should be and whether it should be a moving coordinate system. The answer to the question derives from a compromise between two conflicting issues:

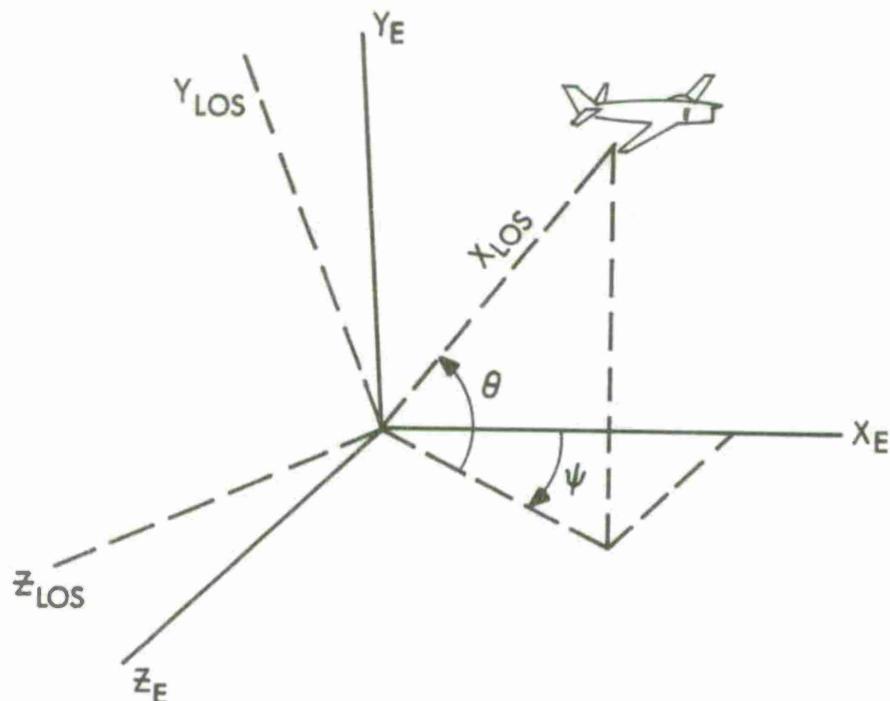
1. Computational convenience
2. Ease of representation of the noise vectors.

In the latter issue, one is faced with the reality that target input noise is most conveniently represented in a coordinate system closely tied to the target. On the other hand, measurement noise is most conveniently represented in a coordinate system which is tied to a

sensor. In one case Cartesian coordinates are called for, and in the other case spherical polar coordinates are called for. These issues are further complicated by the fact that the equations of motion are written with respect to an inertial coordinate system which may not be one of the target or sensor coordinate system.

2.1 MOVING COORDINATE FRAMES

The purpose of this section is to clarify the relationship between the various possible coordinate systems which one might employ and to propose a convenient coordinate frame. Consider first, the inertia coordinate system E, and the moving coordinate system, LOS (line of sight).



At any instant, any vector in one of these systems can be represented in the other system through transformation, \underline{T}_{SE} .

$$\underline{x}_E = \underline{T}_{SE} \underline{x}_{LOS} \quad \text{position} \quad 2.1$$

$$\underline{v}_E = \underline{T}_{SE} \underline{v}_{LOS} \quad \text{velocity} \quad 2.2$$

$$\underline{a}_E = \underline{T}_{SE} \underline{a}_{LOS} \quad \text{acceleration} \quad 2.3$$

$$\underline{T}_{SE} = \begin{bmatrix} \cos\theta & \cos\Psi & -\sin\theta & \cos\Psi & -\sin\Psi \\ \sin\theta & & \cos\theta & & 0 \\ \cos\theta & \sin\Psi & -\sin\theta & \sin\Psi & \cos\Psi \end{bmatrix} \quad 2.4$$

The equations of motion provide the relationships between these vectors.

$$\dot{\underline{x}}_E = \underline{v}_E \quad 2.5$$

$$\dot{\underline{v}}_E = \underline{a}_E \quad 2.6$$

$$\ddot{\underline{a}}_E = -\alpha \underline{a}_E + \underline{U}_E \quad 2.7$$

From the transformations, the derivatives may also be written

$$\dot{\underline{x}}_E = \dot{\underline{T}}_{SE} \underline{x}_{LOS} + \underline{T}_{SE} \dot{\underline{x}}_{LOS} \quad 2.8$$

$$\dot{\underline{v}}_E = \dot{\underline{T}}_{SE} \underline{v}_{LOS} + \underline{T}_{SE} \dot{\underline{v}}_{LOS} \quad 2.9$$

$$\dot{\underline{a}}_E = \dot{\underline{T}}_{SE} \underline{a}_{LOS} + \underline{T}_{SE} \dot{\underline{a}}_{LOS} \quad 2.10$$

Combining these with the equations of motion, one obtains the equations of motion in a moving coordinate system.

$$\dot{\underline{x}}_{LOS} = -Q^* \underline{x}_{LOS} + \underline{v}_{LOS} \quad 2.11$$

$$\dot{\underline{v}}_{\text{LOS}} = -\underline{Q}^* \underline{v}_{\text{LOS}} + \underline{a}_{\text{LOS}} \quad 2.12$$

$$\dot{\underline{a}}_{\text{LOS}} = -(\underline{Q}^* + \alpha I) \underline{a}_{\text{LOS}} + \underline{u}_{\text{LOS}} \quad 2.13$$

where

$$\underline{Q}^* = \underline{T}_{\text{se}}^{-1} \dot{\underline{T}}_{\text{se}} = \begin{bmatrix} 0 & -\dot{\theta} & \dot{\psi} \cos \theta \\ \dot{\theta} & 0 & \dot{\psi} \sin \theta \\ \dot{\psi} \cos \theta & \dot{\psi} \sin \theta & 0 \end{bmatrix} \quad 2.14$$

The equations of motion as in Equations 2.11, 2.12, and 2.13 are complete and in every way state the same behavior as in Equations 2.5, 2.6, and 2.7. In the moving coordinate system, LOS, the x axis always remains aligned with the target. This is the approximate alignment of the sensor, thus one might find it convenient to express measurement noise in such a coordinate system. In particular, the velocity in the LOS system can be converted to spherical polar coordinates by

$$\underline{v}_r = \underline{M} \underline{v}_{\text{LOS}} \quad 2.15$$

$$\underline{v}_r = \begin{bmatrix} \dot{r} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = \dot{\underline{x}}_r \quad 2.16$$

$$\underline{M} = \begin{bmatrix} 1 & & \\ & 1/r & \\ & & \frac{1}{r \cos \theta} \end{bmatrix} \quad 2.17$$

In the equations of motion, Equation 2.15 may be used in place of Equation 2.11 to provide a position in spherical polar coordinates.

$$\dot{\underline{x}}_r = \underline{M} \underline{v}_{\text{LOS}} \quad 2.18$$

$$\dot{\underline{v}}_{\text{LOS}} = -\underline{Q}^* \underline{v}_{\text{LOS}} + \underline{a}_{\text{LOS}} \quad 2.19$$

$$\dot{\underline{a}}_{\text{LOS}} = -(\underline{Q}^* + \alpha I) \underline{a}_{\text{LOS}} + \underline{u}_{\text{LOS}} \quad 2.20$$

and the observations may be expressed as

$$\underline{Y} = \underline{H} \underline{x}_r + \underline{V} \quad 2.21$$

where \underline{V} is a measurement noise expressed in spherical polar coordinates. Equations 2.18, 2.19, and 2.20 are once again complete equations of motion in that they express the same target behavior as Equations 2.5, 2.6 and 2.7. These new equations are nonlinear, and must be subjected to linearization prior to use as a basis for design of a Kalman estimator. The matrices, \underline{M} and \underline{Q}^* , are the sources of the nonlinearity since they are functions of the components of \underline{x}_r .

Use of the latest estimates of θ and ψ provides one possible linearization for \underline{M} , \underline{Q}^*

$$\underline{M}(r, \theta) \approx \underline{M}(\hat{r}, \hat{\theta}) \quad 2.22$$

$$\underline{Q}^*(\theta, \psi) \approx \underline{Q}^*(\hat{\theta}, \hat{\psi}) \quad 2.23$$

2.2 STEPPING COORDINATES

In the case of \underline{Q}_k^* an alternate approach is often taken which is known as "stepping coordinates." In such an approach it is assumed that T_{SE} is constant over a small time period Δt , i.e. from the k^{th} to the $k+1^{\text{th}}$ observation ,

$$\dot{\underline{T}}_{SE} = \underline{0} \quad 2.24$$

$$\dot{\underline{Q}}^* = \underline{0} \quad 2.25$$

which leads to equations of motion in stepping coordinates.

$$\dot{\underline{x}}_r = \underline{M}(\hat{r}, \hat{\theta}) \underline{v}_{LOS} \quad 2.26$$

$$\dot{\underline{v}}_{LOS} = \underline{a}_{LOS} \quad 2.27$$

$$\dot{\underline{a}}_{LOS} = -\alpha \underline{a}_{LOS} + \underline{U}_{LOS} \quad 2.28$$

If the vectors \underline{v}_{LOS} and \underline{a}_{LOS} are in the correct coordinate system at time t , they will not be at time $t + \Delta t$ because the coordinate system has moved. Once the vectors are obtained at $t + \Delta t$, the coordinate system is "stepped" to its new position and the vectors expressed in this new coordinate system. The size of the step is determined by \underline{x}_r . The change of the coordinate transformation, \underline{T}_{SE} , may be expressed as a Taylor's series in its previous value

$$\underline{T}_{SE}^{k+1} = \underline{T}_{SE}^k + \dot{\underline{T}}_{SE}^k \Delta t + \dots \quad 2.29$$

Using only the first two terms as an approximation.

$$(\underline{T}_{SE}^{k+1})^T \underline{T}_{SE}^k \approx \underline{I} - \dot{\underline{Q}}^* \Delta t \quad 2.30$$

This transformation will convert the velocities and accelerations to the new coordinate system after time, Δt .

$$\underline{a}_{LOS}^{k+1}(t + \Delta t) = (\underline{T}_{SE}^{k+1})^T \underline{T}_{SE}^k \underline{a}_{LOS}^k(t + \Delta t) \quad 2.31$$

$$\underline{v}_{LOS}^{k+1}(t + \Delta t) = (\underline{T}_{SE}^{k+1})^T \underline{T}_{SE}^k \underline{v}_{LOS}^k(t + \Delta t) \quad 2.32$$

where the superscripts refer to the particular instance of an LOS coordinate system. Further approximations on Equations 2.30 lead to

$$(\underline{T}_{SE}^{k+1})^T \underline{T}_{SE}^k = \begin{bmatrix} 1 & \hat{\Delta\theta} & \hat{\Delta\Psi}\cos\hat{\theta} \\ -\hat{\Delta\theta} & 1 & \hat{\Delta\Psi}\sin\hat{\theta} \\ \hat{\Delta\Psi}\cos\hat{\theta} & \hat{\Delta\Psi}\sin\hat{\theta} & 1 \end{bmatrix} \quad 2.33$$

Thus Equations 2.26, 2.27, 2.28, 2.31, 2.32 and 2.33 are linearized equations of motion expressed in stepping coordinates.

The target noise vector, \underline{U}_{LOS} , in Equation 2.28 remains in a somewhat inconvenient coordinate system. It is most conveniently expressed in a coordinate system aligned with the velocity vector of the target. Such a noise vector, \underline{U}_T , may be transformed to the LOS system by

$$\underline{U}_{LOS} = \underline{T}_{SE}^T \underline{T}_{TE} \underline{U}_T \quad 2.34$$

where \underline{T}_{TE} depends upon the angular orientation of the velocity vector of the target. The transformation, \underline{T}_{TE} , is derived from the velocity vector, \underline{v}_{LOS} , by

$$1) \quad \underline{v}_E = \underline{T}_{SE} \underline{v}_{LOS} \quad 2.35$$

$$2) \quad \theta_v = \tan^{-1} \frac{(v_y)_E}{((v_x)_E^2 + (v_z)_E^2)^{1/2}} \quad 2.36$$

$$3) \quad \Psi_v = \tan^{-1} \frac{(v_z)_E}{(v_x)_E} \quad 2.37$$

3) \underline{T}_{TE} is the same as \underline{T}_{SE} (equation 2.4) with θ , and Ψ replaced by θ_v and Ψ_v respectively

2.3 A Linearized Target and Observation Model

The derivations given in the previous section lead to a linearized target model with the observation error represented in spherical polar

coordinates and with the target noise expressed in a target velocity oriented Cartesian frame. The resulting model may be summarized as

I. Target Model

(a) Position

$$\dot{\underline{x}}_r = \underline{M} (\hat{r}, \hat{\theta}) \underline{v}_{\text{LOS}}; \underline{x}_r = \begin{bmatrix} r \\ \theta \\ \psi \end{bmatrix} \quad 2.38$$

(b) Velocity

$$\underline{v}_{\text{LOS}} = \underline{a}_{\text{LOS}} \quad (\text{Cartesian Coordinates}) \quad 2.39$$

(c) Acceleration

$$\underline{a}_{\text{LOS}} = -\alpha \underline{a}_{\text{LOS}} + \underline{U}_{\text{LOS}} \quad (\text{Cartesian Coordinates}) \quad 2.40$$

II. Observation Model

$$\underline{Y}_r = \underline{H} \underline{x}_r + \underline{V}_r \quad (\text{Spherical Polar Coordinates}) \quad 2.41$$

III. Target Noise

$$\underline{U}_{\text{LOS}} = \underline{T}_{\text{SE}}^T \underline{T}_{\text{TE}} \underline{U}_T; \underline{U}_T = \begin{bmatrix} U \\ U_x \\ U_y \\ U_z \end{bmatrix} \quad 2.42$$

where: U_x is along the velocity vector of the target.

U_z is in the horizontal plane.

U_y completes the orthogonal set of components.

IV. Stepping Coordinates

$(\underline{T}_{\text{SE}}^{k+1})^T (\underline{T}_{\text{SE}}^k)$ \equiv Transformation from the LOS frame at time k , to the LOS frame at time, $k+1$ (Equation 2.33). 2.43

Note that this model has the position vector in spherical polar coordinates and the velocity and acceleration vectors in cartesian coordinates. There is a linearization required in Equation 2.35, but since M is diagonal, only three components need to be approximated. The stepping coordinate transformation may be performed analytically, but experience gathered so far suggests that the approximation shown in Equation 2.33 is quite adequate. However, no linearization is required to step the coordinates.

The above formulation of a linearized model is only one of a number of possibilities. As an alternative the entire target model may be expressed in spherical polar coordinates. In such a formulation the target model equations are still nonlinear, i.e. quadratic terms in angular rates, and the target noise must be subjected to another transformation to convert it to spherical polar coordinates.

As a third possibility, if the entire target model is expressed in Cartesian coordinates, then the observation model is nonlinear. The perceived observations are still, of course, in polar coordinates. The observations cannot be linearized by approximations of the same type already suggested in the proposed target model. The first term of the observation model, i.e., \underline{Hx} when linearized, is inherently nonlinear in the sense that a cartesian position vector does not factor out. The linearization will require a Taylor series approximation of the \underline{Hx} term for which the choice of the origin of the expansion is crucial to obtaining a useable value of \underline{H} . The matrix, \underline{H} , plays an important role in the estimation algorithms which follow. Thus a Cartesian position vector is to be avoided if at all possible.

It would appear that the simplest compromise between target noise transformations and polar position vectors would be to leave the positions vectors (possibly the velocities also) in spherical polar coordinate and represent the accelerations in Cartesian coordinates. The resulting hybrid equations of motion were summarized earlier in this section.

In order to explicitly show the forms of the various alternative linearizations, the position, velocity, and acceleration vectors will be combined into a single state vector, \underline{X} , which in the various coordinate frames becomes:

(a) Spherical Polar

$$\underline{X} = \begin{bmatrix} r \\ v \\ r \\ a \\ x \\ \theta \\ \omega \\ z \\ a \\ y \\ \psi \\ \omega \\ w \\ y \\ a \\ z \end{bmatrix} \quad 2.44$$

$$\underline{a}_{\text{LOS}} = \begin{bmatrix} a \\ x \\ a \\ y \\ a \\ z \end{bmatrix}$$

where: ω_z = Angular rate about the Z_{LOS} axis.

ω_y = Angular rate about the Y_{LOS} axis.

(b) Cartesian

$$\underline{\dot{X}} =$$

$$\begin{bmatrix} x \\ v_x \\ a_x \\ x \\ Y \\ v_y \\ a_y \\ y \\ z \\ v_z \\ a_z \\ z \end{bmatrix}$$

2.45

(c) Hybrid

$$\underline{\dot{X}} =$$

$$\begin{bmatrix} r \\ v_r \\ x \\ a_x \\ \theta \\ v_\theta \\ y \\ a_y \\ \psi \\ v_\psi \\ z \\ a_z \\ z \end{bmatrix}$$

2.46

The linearized model in each case is of the form

$$\underline{\dot{X}} = \underline{A} \underline{X} + \underline{B} \underline{U}$$

2.47

where the parameters A, and B for the various coordinates frames are:

(a) Spherical Polar *

$$\underline{A} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -\hat{\omega}_z & 0 & 0 & -\hat{\omega}_y & 0 \\ 0 & 0 & -\alpha & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{2\hat{v}_r}{\hat{r}} & \frac{1}{\hat{r}} & 0 & -\tan\theta\hat{\omega}_y \\ 0 & 0 & 0 & 0 & 0 & 0 & -\alpha & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\hat{r}\cos\theta} & 0 \\ 0 & 0 & 0 & 0 & 0 & \tan\theta\hat{\omega}_y & 0 & 0 & \frac{1}{\hat{r}} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\alpha \end{bmatrix} \quad 2.48$$

$$\underline{B} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

* This form leaves the accelerations in the LOS coordinate frame and assumes that stepping coordinates are employed for the accelerations.

Note the presence of quadratic terms in the angular rates

(b) Cartesian

$$\underline{A} = \begin{bmatrix} 0 & 1 & 0 & & \\ 0 & 0 & 1 & \underline{0} & \underline{0} \\ 0 & 0 & -\alpha & & \\ & & 0 & 1 & 0 \\ & \underline{0} & 0 & 0 & 1 \\ & & 0 & 0 & -\alpha \\ & & \underline{0} & 0 & 1 \\ & & & 0 & 1 & 0 \\ & & & & 0 & 0 & 1 \\ & & & & 0 & 0 & -\alpha \end{bmatrix} \quad \underline{B} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

2.49

(c) Hybrid

$$\underline{A} = \begin{bmatrix} 0 & 1 & 0 & & \\ 0 & 0 & 1 & \underline{0} & \underline{0} \\ 0 & 0 & -\alpha & & \\ & & 0 & \frac{1}{r} & 0 \\ & \underline{0} & 0 & 0 & 1 \\ & & 0 & 0 & -\alpha \\ & & \underline{0} & 0 & 1 \\ & & & 0 & \frac{1}{r \cos \theta} \\ & & & & 0 \end{bmatrix} \quad \underline{B} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

2.50

The observation models also show the influence of the choice of coordinate frames. The observation models are of the general form.

$$\underline{Y} = \underline{H} \underline{X} + \underline{V} \quad 2.51$$

where in each case:

(a) Spherical Polar and Hybrid

$$\underline{H} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \quad 2.52$$

(c) Cartesian: The observation model in this case is derived from Taylor series expansion of

$$x = r \cos\theta \cos\psi$$

$$y = r \sin\theta \quad 2.53$$

$$z = r \cos\theta \sin\psi$$

leading to the form

$$\underline{x}_r = \underline{x}_r(0) + \underline{H}(\underline{x} - \underline{x}(0)) \quad 2.54$$

where $\underline{x}_r(0)$ and $\underline{x}(0)$ are the origin of the expansion and are conveniently chosen to be

$$\underline{x}_r(0) = (\hat{\underline{x}}_{r,k})_{k,k-1} \quad \text{spherical polar position} \quad 2.55$$

$$\underline{x}(0) = (\hat{\underline{x}}_{k,k-1}) \quad \text{cartesian position} \quad 2.56$$

Thus a fictitious observation is formed

$$\underline{Y} - (\hat{\underline{X}}_r)_{k,k-1} + \underline{H} \hat{\underline{X}}_{k,k-1} = \underline{H} \underline{X} + \underline{V} \quad 2.57$$

but the innovation process simplifies to

$$\underline{Y} - (\hat{\underline{X}}_r)_{k,k-1} \quad 2.58$$

In this case the H matrix becomes

$$\underline{H} = \begin{bmatrix} \cos\hat{\theta} \cos\hat{\psi} & 0 & 0 & \sin\hat{\theta} & 0 & 0 & \cos\hat{\theta} & \sin\hat{\psi} & 0 & 0 \\ -\sin^2\hat{\theta} \cos\hat{\psi} & 0 & 0 & \sin\hat{\theta} \cos\hat{\theta} & 0 & 0 & -\sin^2\hat{\theta} \sin\hat{\psi} & 0 & 0 \\ y & y & y & y & y & y & y & y & y & y \\ -\sin^2\hat{\psi} & 0 & 0 & 0 & 0 & 0 & -\sin\hat{\psi} & \cos\hat{\psi} & 0 & 0 \\ z & z & z & z & z & z & z & z & z & z \end{bmatrix} \quad 2.59$$

This illustrates the extent to which approximations must be employed to linearize the observation model.

In this case, the raw observations could also be converted to Cartesian coordinates by Equation 2.53. Using the original observation equation, this transformation may be represented as Equation 2.61.

$$\underline{Y}_r = \underline{X}_r + \underline{V}_r \quad (\text{Position observations in Spherical Polar Coordinates}) \quad 2.60$$

$$\underline{Y} = f(\underline{Y}_r) = f(\underline{X}_r + \underline{V}_r) \quad (\text{Position observations converted to Cartesian Coordinates by } f(\underline{Y}_r).) \quad 2.61$$

Using a Taylor series approximation to the right hand side.

$$\underline{Y} = f(\underline{X}_r) + \left. \frac{\partial f}{\partial \underline{X}_r} \right|_{\underline{X}_r} \underline{V}_r + \dots \quad 2.62$$

or, using only the first two terms as an approximation

$$\underline{Y} = \underline{X} + T_{rE} \frac{\underline{V}}{r} \quad 2.63$$

where

$$T_{rE} = \begin{bmatrix} \hat{\cos\theta} & \hat{\cos\psi} & -\hat{r}\sin\theta & \hat{\cos\psi} & -r & \hat{\cos\theta} & \hat{\sin\psi} \\ \hat{\sin\theta} & \hat{\cos\psi} & \hat{r} & \hat{\cos\theta} & 0 & & \\ \hat{\cos\theta} & \hat{\sin\psi} & -r & \hat{\sin\theta} & \hat{\sin\psi} & \hat{r} & \hat{\cos\theta} & \hat{\cos\psi} \end{bmatrix} \quad 2.64$$

This transformation appears most prominently in the representation of the observation noise statistics.

$$R_E = T_{rE} R_r T_{rE}^T : \text{Covariance of the observation noise in Cartesian Coordinates} \quad 2.65$$

The innovation process would be

$$\underline{f}(\underline{Y}_r) - H \hat{\underline{X}}_{k,k-1} \quad 2.66$$

where H is of the form

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \quad 2.67$$

It has been shown in this discussion that the preferred coordinate systems are

1. Cartesian Coordinates for the Target Model.
2. Spherical Polar Coordinates for the Observation Model.

The advantages of the hybrid coordinate system should now be evident. The best of both possible approaches is achieved by representing the observed components of the target state in spherical polar coordinates,

and the remaining components in Cartesian coordinates. The linearizations in the hybrid approach are as few and as straightforward as one could hope to achieve in this paradoxical situation.

3.0 Estimation Algorithms

3.1 The Kalman Estimator

Most of the latest generation of fire control systems utilize an estimation algorithm proposed by Kalman ⁽⁴⁾ and discussed in numerous texts (1) (2) (3). The Kalman algorithm, also known as the Kalman filter, provides an extension of least squares estimation techniques to linear dynamical systems. Given a discrete dynamical system of the form;

$$\underline{x}_{k+1} = \underline{\Phi}_k \underline{x}_k + \underline{B}_k \underline{U}_k \quad (\text{Equation 1.6}) \quad 3.1$$

together with an observation model,

$$\underline{Y}_k = \underline{H}_k \underline{x}_k + \underline{V}_k \quad (\text{Equation 1.7}) \quad 3.2$$

and noise models

$$\underline{Q}_k = E(\underline{U}_k \underline{U}_k^T) \quad (\text{Equation 1.8}) \quad 3.3$$

$$\underline{R}_k = E(\underline{V}_k \underline{V}_k^T) \quad (\text{Equation 1.9}) \quad 3.4$$

The purpose of the estimation algorithm is to obtain estimates, $\hat{\underline{x}}_k$, of the state which are as close as possible to the true state, \underline{x}_k , and which are unbiased in the sense that,

$$E(\hat{\underline{x}}_k) = E(\underline{x}_k) \quad 3.5$$

The measure of closeness which is utilized is the norm of

$$\underline{P}_k \equiv ((\hat{\underline{x}}_k - \underline{x}_k) (\hat{\underline{x}}_k - \underline{x}_k)^T) \quad 3.6$$

This matrix represents the variance of the estimation errors.

The Kalman algorithm suggests the computation of a predicted estimate, $\hat{x}_{k,k-1}$ (The estimate based upon k-1 observations) and a corrected estimate, $\hat{x}_{k,k}$ (The estimate based upon k observation).

The associated error variances are written as

$$\underline{P}_{k,k-1} \text{ and } \underline{P}_{k,k}$$

The predictor phase of the estimation is

$$\hat{x}_{k,k-1} = \hat{\Phi} \hat{x}_{k-1,k-1} \quad 3.7$$

$$\underline{P}_{k,k-1} = \underline{\Phi} \underline{P}_{k-1,k-1} \underline{\Phi}^T + \underline{B}_k \underline{Q}_k \underline{B}_k^T \quad 3.8$$

and the corrected estimate is

$$\hat{x}_{k,k} = \hat{x}_{k,k-1} + K_k (Y_k - H_k \hat{x}_{k,k-1}) \quad 3.9$$

$$\underline{P}_{k,k} = \underline{P}_{k,k-1} - \underline{P}_{k,k-1} \underline{H}_k^T (\underline{R}_k + \underline{H}_k \underline{P}_{k,k-1} \underline{H}_k^T)^{-1} \underline{H}_k \underline{P}_{k,k-1} \quad 3.10$$

$$K_k = \underline{P}_{k,k-1} \underline{H}_k^T (\underline{R}_k + \underline{H}_k \underline{P}_{k,k-1} \underline{H}_k^T)^{-1}$$

where K_k is often referred to as the Kalman gain.

The equations 3.7 through 3.10 define a recursive computation of a sequence of estimates based upon a sequence of observations, Y_k , which arrive one at a time. The variances of the estimation errors are computed routinely along with the estimates. It should be noted that Equations 3.8, 3.9, and 3.10 represent an independent sequence of computations. In particular, for a pre-specified \underline{Q}_k and \underline{R}_k , the Kalman gain can be computed independently of the observations, Y_k .

In situations where the theoretical model is known to only partially represent the true target behavior, i.e. during maneuvers, the physical significance of α (Equation 1.3), and Q_k may be unattainable. A more algorithmic view would lead one to choose α and Q_k in such a way that the estimates have desirable properties in terms of frequency content or predicting power.

3.2 Prediction

In its purest form the process of predicting the future position of the target involves integrating the equations of motion over the time-of-flight of the projectile. However, rather than resort to numerical integration, the same linearizations and discretizations employed in the Kalman estimator may be employed to give a predictor of the form.

$$\hat{\underline{x}}(t+t_f) = \underline{\Phi}(t, t_f) \hat{\underline{x}}(t) \quad 3.11$$

where t_f is the projectile's time of flight.

Since observations are not involved in this process, it is most convenient and accurate to perform the predictions in Cartesian coordinates. Furthermore, if only the positions are to be predicted, only the inner product of the top view of $\underline{\Phi}$ and \underline{x} needs to be computed.

3.3 Biased Estimation

It has been shown that biased estimates have some highly desirable properties in both least squares and dynamical problems. A biased estimation scheme based upon the work of Hoerl⁽⁵⁾ and Chang⁽⁶⁾ is outlined here as a suggested refinement to the Kalman filter.

The biased estimation algorithm requires obtaining the unbiased estimate, $\hat{\underline{x}}_{k,k-1}$, and $\hat{\underline{x}}_{k,k}$, according to equations 3.6 and 3.8. The "corrected estimate", $\hat{\underline{x}}_{k,k}^*$ is further refined by

$$\hat{\underline{x}}_{k,k}^* = \underline{z} \hat{\underline{x}}_{k,k}; \quad 3.12$$

where

$$\underline{Z} = (\underline{I}_p - K \underline{P}_{k,k} \underline{\alpha} - (\underline{K} \underline{\alpha}^T \underline{P}_{k,k} \underline{\alpha} + \underline{I}_q)^{-1} \underline{\alpha}^T) \quad 3.13$$

$\hat{\underline{x}}_{k,k}^*$ is a biased estimate

K is a positive scalar;

$\underline{\alpha}$ is a $p \times q$ matrix of rank, q , and p is the dimension of \underline{x}_k .

Using the following definitions of variances

$$\underline{P}_{k,k} \equiv E((\hat{\underline{x}}_{k,k} - \underline{x}_k)(\hat{\underline{x}}_{k,k} - \underline{x}_k)^T) \quad 3.14$$

$$\underline{P}_{k,k}^* \equiv E((\hat{\underline{x}}_{k,k}^* - \underline{x}_k)(\hat{\underline{x}}_{k,k}^* - \underline{x}_k)^T) \quad 3.15$$

$$\underline{G}_{k,k} \equiv E((\hat{\underline{x}}_{k,k} - E(\underline{x}_k))(\hat{\underline{x}}_{k,k} - E(\underline{x}_k))^T) \quad 3.16$$

$$\underline{G}_{k,k}^* \equiv E((\hat{\underline{x}}_{k,k}^* - E(\underline{x}_k))(\hat{\underline{x}}_{k,k}^* - E(\underline{x}_k))^T) \quad 3.17$$

it can be shown that the biased estimates have the properties

$$\underline{P}_{k,k}^* \geq \underline{P}_{k,k} \quad 3.18$$

$$\underline{G}_{k,k}^* \leq \underline{G}_{k,k} \quad 3.19$$

and further that for small K

$$\underline{P}_{k,k}^* - \underline{P}_{k,k} = \underline{0}(K^2) \quad 3.20$$

$$\underline{G}_{k,k}^* - \underline{G}_{k,k} = \underline{0}(K) \quad 3.21$$

This indicates that the loss of "optimality" in the Kalman sense is small while the shift of the estimate toward the expectation of the true state may be significant. This should have a smoothing effect upon the estimates, and thus make the estimates more suitable for prediction.

The biased estimate (Equation 3.12) involves a matrix, \underline{Z} , which contains two parameters, K and $\underline{\alpha}$. These parameters are subject to choice on the basis of computational convenience. One possibility

is to choose $\underline{\alpha}$ such that

$$\underline{\alpha} = \underline{I}_p \quad 3.22$$

then

$$\underline{z} = (\underline{I}_p + K \underline{P}_{k,k})^{-1} \underline{I}_p - K \underline{P}_{k,k} \quad 3.23$$

Using this approximation, the biased estimate becomes

$$\hat{\underline{x}}_{k,k}^* = (\underline{I}_p - K \underline{P}_{k,k}) \hat{\underline{x}}_{k,k} \quad 3.24$$

This form has the advantage of placing the greatest bias upon those components of estimate which have the largest variances. The components which are least certain will be shifted the most.

4. Trajectory Algorithms

4.1 Deterministic Trajectories

The target modeling techniques discussed in the previous sections will be extended to provide descriptions of actual tactical trajectories. These trajectories are invaluable to the simulation of a fire control system, and to the design of the essential estimation and prediction algorithms.

The simplest trajectory segment to simulate is the straight line. The model requires a fixed velocity; thus a state transition of the form.

$$\underline{x}_{k+1} = \Phi_k \underline{x}_{k+1}; \quad 4.1$$

where all components are in Cartesian coordinates.

$$\underline{\Phi} = \begin{bmatrix} 1 & \Delta t & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \Delta t \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad 4.2$$

$$\underline{x}_k = \begin{bmatrix} x \\ v \\ x \\ y \\ v \\ y \\ z \\ v \\ z \end{bmatrix}_k \quad 4.3$$

Given an initial position and velocity, \underline{x}_0 , this model, (Equation 4.1), can be used to propagate the position forward in time along a straight line. Each step requires a matrix by vector multiplication.

In the case of circular motion , i.e. constant magnitude of acceleration , a similar model is possible.

$$\underline{x}_{k+1}^c = \underline{\Phi}_k \underline{x}_k^c \quad 4.4$$

where \underline{x}_k^c is same as Equation 4.3 except that the position components are represented with respect to center of curvature of circle. This new position vector is related to the old one by

$$\underline{x}_E = \underline{x}_L + \underline{x}_E^c \quad (\text{position only}) \quad 4.5$$

where \underline{x}_L is the position of the center of curvature, and \underline{x}_E is the position in the inertial or earth frame of reference. The state transition matrix, $\underline{\Phi}_k$, is now

$$\underline{\phi}_k = \begin{bmatrix} \cos -\omega\Delta t & \frac{1}{\omega} \sin \omega\Delta t & & & \\ -\omega \sin \omega\Delta t & \cos \omega\Delta t & & & 0 \\ & & 0 & & 0 \\ & & & \cos \omega\Delta t & \frac{1}{\omega} \sin \omega\Delta t \\ & & & -\omega \sin \omega\Delta t & \cos \omega\Delta t \\ & & & & 0 \\ & & & & & \cos \omega\Delta t & \frac{1}{\omega} \sin \omega\Delta t \\ & & & & & -\omega \sin \omega\Delta t & \cos \omega\Delta t \\ & & & & & & 4.6 \end{bmatrix}$$

Where ω is the angular rate of circular motion. The parameters, ω , \underline{x}_o and \underline{x}_L , must be specified to determine a circular trajectory. If the circular trajectory is only one segment of a continuous smooth trajectory, then the initial position and velocity are present. The radius of curvature is also constrained to be perpendicular to the velocity vector. All of these constraints leave only three degrees of freedom in specifying a circular trajectory:

- 1.) ω - The angular rate.
- 2.) The magnitude of the radius of curvature.
- 3.) The orientation of the radius of curvature (one angle)

In a velocity vector oriented coordinate system, i.e. X axis is aligned with the velocity vector, the position vector with respect to the center of curvature is

$$(\underline{x}_v^c) = -r^c \begin{bmatrix} 0 \\ \cos \theta_c \\ \sin \theta_c \end{bmatrix} \quad 4.7$$

where θ_c is the orientation of the plane of motion with respect to the vertical plane, and r^c is the magnitude of the radius of curvature. The position of the center of curvature is therefore

$$\underline{x}_L = (\underline{x})_E - T_{vE} (\underline{x}_v^c) \quad 4.8$$

and the initial position vector with respect to the center of curvature is

$$\underline{x}_0^c = T_{-\underline{v}E} \underline{x}_v^c \quad 4.9$$

Thus, for circular motion, the positions and velocities may be propagated forward in time by the following sequence of computations:

1. Initialization

- (a) Given $(\underline{x}_0)_E$ and $(\underline{v}_0)_E$ with respect to a convenient inertial coordinate system.
- (b) Specify ω , r^c , θ_c .
- (c) Compute and hold \underline{x}_L (Equation 4.7 and 4.8)
- (d) Initialize, \underline{x}^c ; (Equation 4.4)

2. Compute a new state, \underline{x}^c , (Equation 4.4)

3. Compute the state in the inertial coordinate system (Equation 4.5)

4. If trajectory continues, then go to step 2.

5. Result of last step 3 is the current position and velocity.

Trajectories which are segments of either straight lines or circles are easily simulated by the procedure outlined here. Invariance of the state vector between segments will assure continuous, smooth trajectories.

4.2 Stochastic Trajectories.

The equations of the previous section may be extended to allow random motion to be superimposed upon the straight line and circular motion.

In a single component, e.g. the X-component, the equations of motion are:

(a) Straight line trajectory

$$\dot{x} = v_x \quad 4.10$$

$$v_x = a_x; a_x \text{ is a random forcing function.} \quad 4.11$$

(b) Circular trajectory

$$\dot{x} = v_x \quad 4.12$$

$$\dot{v}_x = -x + a_x; a_x \text{ is a random forcing function} \quad 4.13$$

In the case of the straight line, the equations of motion are in the same form as Equations 1.1 and 1.2. The straight line equations lead to an A and D matrix identical to equation 2.46. In the case of circular motion the A matrix becomes

$$\underline{A}^C = \begin{bmatrix} 0 & 1 & 0 & & \\ -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & -\alpha & & \\ & & 0 & 1 & 0 \\ & & 0 & -1 & 0 & 1 & 0 \\ & & & 0 & 0 & -\alpha & & \\ & & & & & 0 & 1 & 0 \\ & & & & & 0 & -1 & 0 & 1 \\ & & & & & 0 & 0 & -\alpha & & \end{bmatrix} \quad 4.14$$

A discrete form of the equations of motion may be obtained by integration over a step Δt . The resulting $\underline{\phi}_k$ and \underline{B}_k are

$$\underline{\phi}_k = \begin{bmatrix} \gamma_{11} & 0 & 0 \\ 0 & \gamma_{22} & 0 \\ 0 & 0 & \gamma_{33} \end{bmatrix} \quad 4.15$$

where

$$\gamma_{ii} = \begin{bmatrix} \cos \omega \Delta t & \frac{1}{\omega} \sin \omega \Delta t & \phi_{23} \\ -\omega \sin \omega \Delta t & \cos \omega \Delta t & \phi_{23} \\ 0 & 0 & e^{-\alpha \Delta t} \end{bmatrix}$$

for $\gamma = 1, 2, 3$.

and

$$\underline{\phi}_{13} = \frac{1}{\omega^2 + \alpha^2} (e^{-\alpha\Delta t} - \cos\omega\Delta t) + \frac{\alpha}{\omega} \sin\omega\Delta t \quad 4.16$$

$$\underline{\phi}_{23} = \frac{1}{\omega^2 + \alpha^2} (\omega \sin\omega\Delta t - (e^{-\alpha\Delta t} - \cos\omega\Delta t)) \quad 4.17$$

and for \underline{B}_k

$$\underline{B}_k = \begin{bmatrix} B_{11} & 0 & 0 \\ B_{21} & 0 & 0 \\ B_{31} & 0 & 0 \\ 0 & B_{11} & 0 \\ 0 & B_{21} & 0 \\ 0 & B_{31} & 0 \\ 0 & 0 & B_{11} \\ 0 & 0 & B_{21} \\ 0 & 0 & B_{31} \end{bmatrix} \quad 4.18$$

where

$$B_{11} = ((\cos\omega\Delta t - 1)/\omega^2) \quad -\phi_{13}) \quad 4.19$$

$$B_{21} = \sin\omega\Delta t / \omega \quad -\phi_{23} \quad 4.20$$

$$B_{21} = (1 - e^{-\alpha\Delta t}) \quad 4.21$$

In the limit as ω goes to zero, these equations become the straight line discrete equations of motion. The equations of motion obtained from Equations 4.15 and 4.18 may be used to simulate circular motion with a random perturbation. Thus, both the deterministic and stochastic aspects of the motion are modeled.

5. Future Work

An Approach to Closed-Loop Systems

Considerable interest has been shown recently in closed-loop fire control systems which use observations of bullet position to correct for modeling errors, drift, misalignment of the gun, etc. The inherent stability and robustness of a feedback configuration may lead to significant improvement in performance of the overall system.

Since the added complexity of a closed-loop system has been difficult to justify a priori, the implementations have taken the configuration of an "add-on" component. The target state estimation and prediction phases of fire control are operated independently of any possible closed-loop configuration. As the justification for closed-loop fire control becomes evident, an overall systems analysis will be needed to properly integrate the closed-loop with the rest of the system. The analyses discussed in this report should be extended to establish the essential mathematical structure and computational burden of such systems.

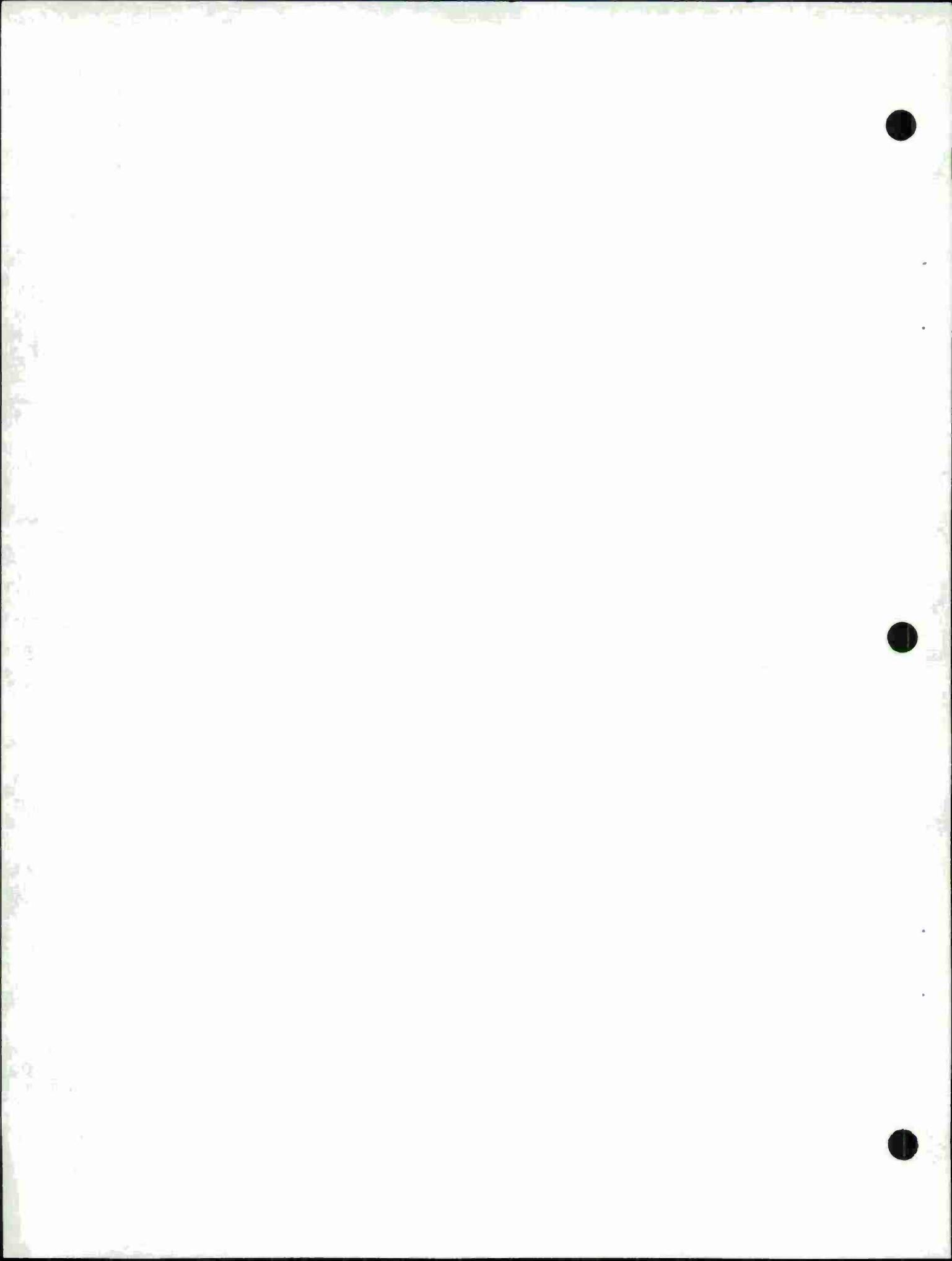
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APPENDIX A

SOFTWARE SUPPORT

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SOFTWARE SUPPORT

In the process of developing the analytic capability to simulate and evaluate fire control systems, a so-called "poly-algorithm" was implemented. This is the software support necessary to set up and run the important algorithms which arise in modern control theory. The design of the software was based upon the premises that:

(1) The algorithms each go through cyclic computations, and there is a point in each cycle where input data may be permitted to influence a computation.

(2) The terminology and notation of modern control theory have stabilized and established the possibility of a finite data base to support the algorithms. The size and entries in the required symbol table may be predetermined without unduly constraining the user's expressive power.

The objective of implementing this poly-algorithm was to provide a simple data handling discipline which would permit the user to select algorithms which would be run singularly or in tandem. Thus, in the application discussed in the body of this report, the trajectory algorithm would be followed by an estimation algorithm which in turn is followed by a prediction algorithm.

The data from one algorithm is used as the parameters for the next algorithm. The user "programs" in a very nonprocedural way in specifying what to do rather than how to do it.

The design of the software was constrained by the requirement that the user be able to escape to a familiar language whenever the poly-algorithm is found to be inadequate. Thus, the entire system was implemented in Fortran. Since some symbolic and character processing is required, some nonstandard features of Fortran are employed but are not permitted to diffuse throughout the programs.

The data and control structure of the poly-algorithm are highly modular in their designs. A hierarchy of control linkages is established according to Figure A-1. The Fortran implementation imposes a strictly imbedded subroutine structure upon the software.

In a typical session of execution of one or more algorithms, the MAIN program passes control to the DRIVER. Data is acquired as necessary to establish parameters for a computation. When the DRIVER encounters a call upon an algorithm, it returns control to MAIN with information about which algorithm is to be initiated (including EXIT if desired). The algorithm will initiate a cycle of the computation during which the DRIVER will be called again. Further data editing is possible while the algorithm is interrupted. The DRIVER passes back to the algorithm information about whether to continue or return to the MAIN program. With control in the MAIN program, the computations continue as though it is the beginning of the session.

The data base established by the poly-algorithm consists of a PRIMARY DATA BASE and a BULK FILE. These form two nodes in a data flow network as shown in Figure A-2. The PRIMARY DATA BASE allocates space on the basis of common names of variables in modern control theory, i.e., X, A, PHI, U, V, etc. The sizes of the arrays are controlled by the user through length variables, i.e., LX, LA, LPHI, LU, LV, etc. The BULK FILES are not associated with any particular variable names, but instead provide storage for time sequences of inputs or outputs of computations. Thus, a time sequence of ranges may be produced by the trajectory algorithms as the X vector, and these same numbers may be retrieved as observations by the Kalman filter in the Y vector.

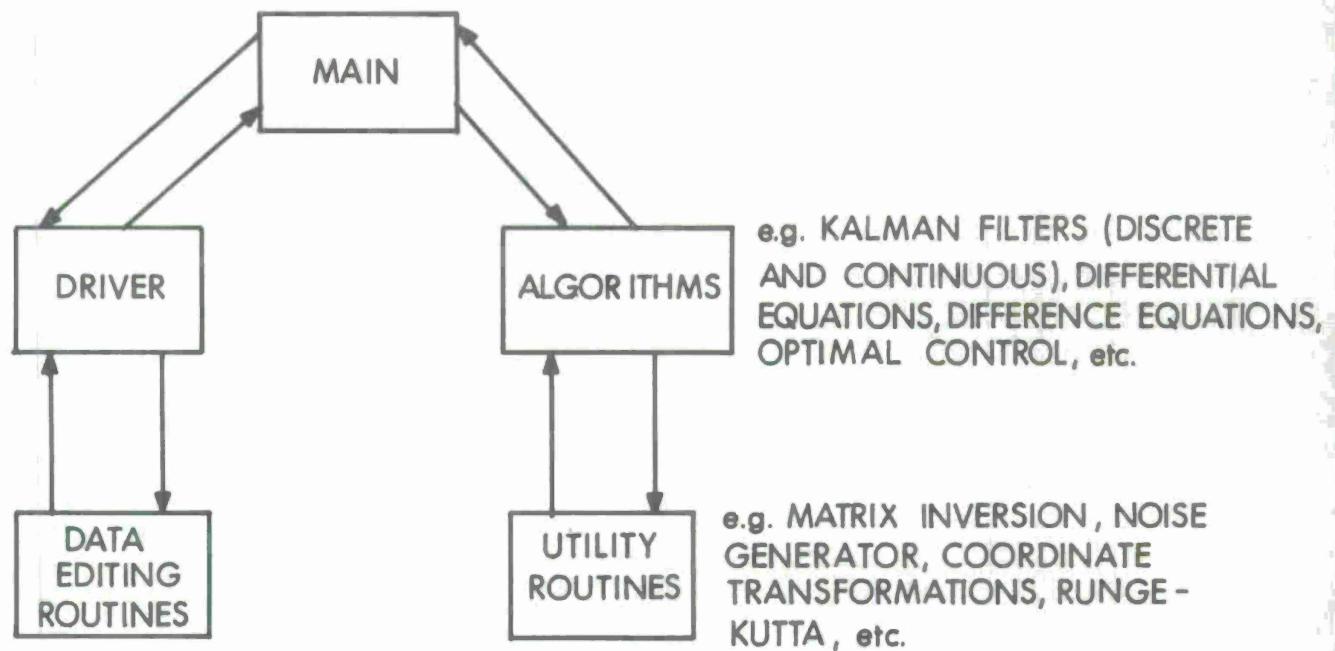


Figure A.1 Control Structure of Modern Control Software.

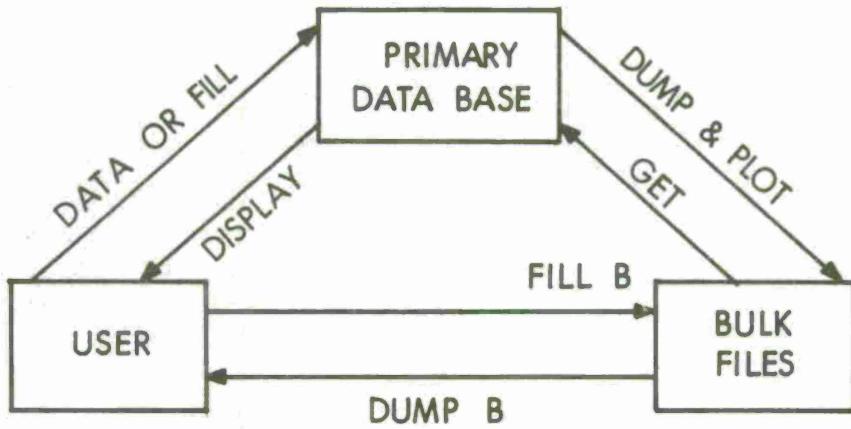
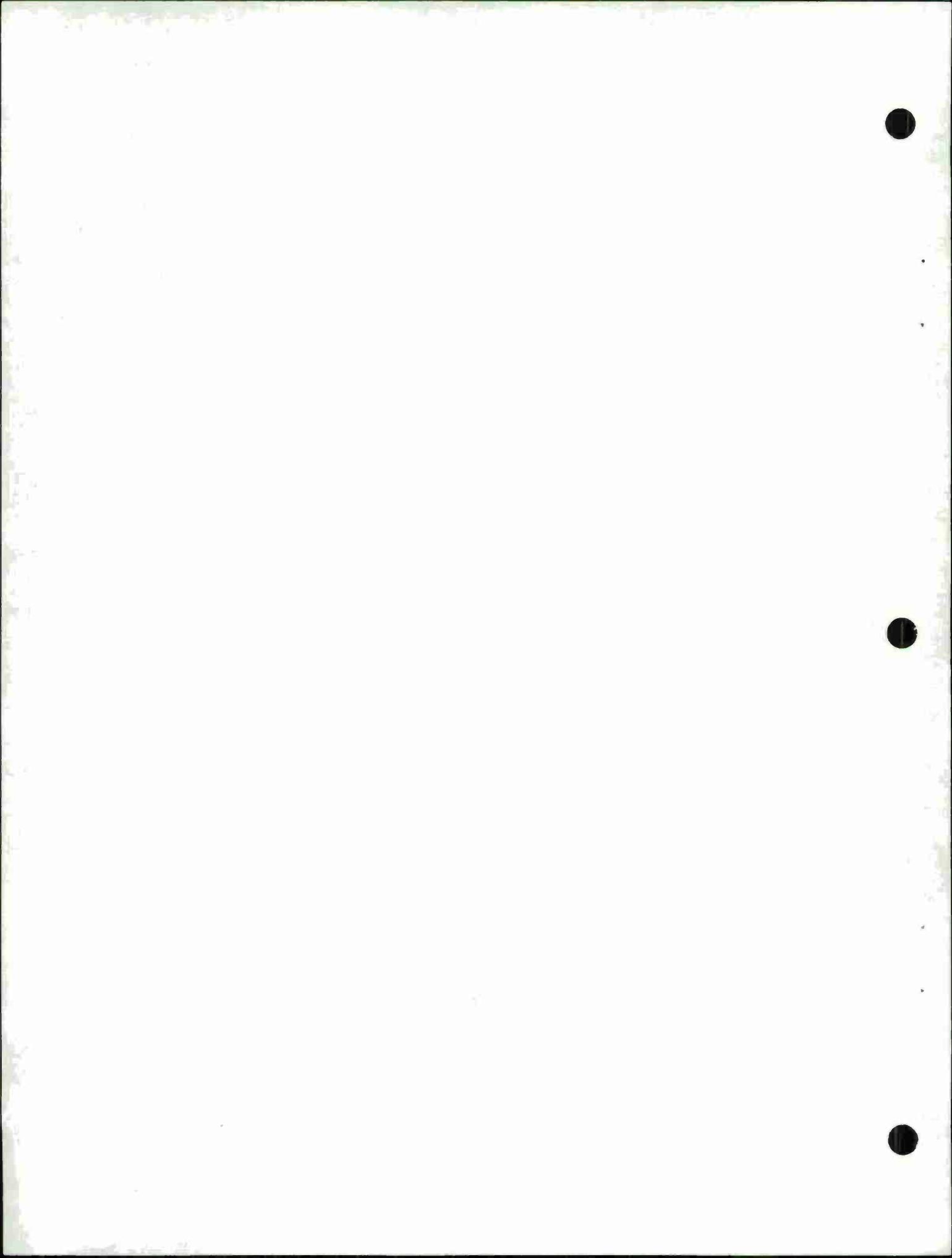


Figure A.2 Data Flow Network for Modern Control Software.

APPENDIX B

SAMPLE FIRE CONTROL COMPUTATIONS

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SAMPLE FIRE CONTROL COMPUTATIONS

As an example of a fire control system computation, the simulation of a target state estimator will be discussed in detail. It will be presumed that the trajectory algorithm has computed positions (r , θ , Ψ) and left them on BULK FILES 8, 9 and 10 respectively. The data set up for the target state estimator would proceed in the following stages.

- a. Specify sizes of all the vectors and matrices.
- b. Generate PHI, B, Q, R and H, DEL (Timestep).
- c. Initialize X, P, and TIME.
- d. Connect BULK FILES 8, 9 and 10 to the observations, Y.
- e. Connect TIME and estimated state, \hat{X} , to BULK FILES in order to save them for prediction.

Having set up the data and established the data flow between the PRIMARY DATA BASE and the BULK FILES, the estimation algorithm is run through as many cycles as desired. During each cycle, data may be displayed as illustrated in Figure B-1.

The data stored by the trajectory algorithm and the estimation algorithm form the basis of a simulation of predictor performance. The estimated states are retrieved, extrapolated ahead in time, and compared with the trajectory. The resulting prediction errors form the basis for evaluation of the fire control system.

A parametric study of the performance of fire control systems is now in preparation and will appear in a forthcoming progress report.

Figure B-1 Example of Kalman Target State Estimator Output

```

TIME      = 0.58000E 01
X          0.31226E 04 -0.24789E 03 -0.19280E 02  0.41842E-01  0.26942E 02
          0.46082E 01  0.15581E 01 -0.29422E 02  0.48014E 02
P          3   3  0.16747E 01
P          6   6  0.11273E 02
P          9   9  0.10266E 02
TIME      = 0.59000E 01
X          0.30996E 04 -0.24732E 03 -0.17745E 02  0.42598E-01  0.27176E 02
          0.44436E 01  0.15570E 01 -0.26652E 02  0.46387E 02
P          a   x           v   y
          a   y           v   z
          a   z           a   z
P          3   y 3  0.15687E 01
P          6   6  0.11240E 02
P          9   9  0.10371E 02
TIME      = 0.60000E 01
X          0.30765E 04 -0.24679E 03 -0.16331E 02  0.43481E-01  0.28014E 02
          0.47462E 01  0.15560E 01 -0.23805E 02  0.44971E 02
P          3   3  0.14715E 01
P          6   6  0.11208E 02
P          9   9  0.10465E 02

```

Estimate of target state, $\hat{\underline{x}} =$

a	x	θ	v
r	y	z	y
a	z		z
ψ	v	a	
3	z	z	

Covariance of accelerations $\sigma_a^2 =$

$$\begin{bmatrix} \sigma_{ax}^2 & & & \\ & \sigma_{ay}^2 & & \\ & & \sigma_{az}^2 & \end{bmatrix}$$

r	- m
v_x	- m/s
a_x	- m/s^2
θ	- rad
v_y	- m/s
a_y	- m/s^2
ψ	- rad
v_z	- m/s
a_z	- m/s^2

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